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KNOWING NUMBERS

Simple arithmetic entails that there are numbers, two of them between 3 and 6, for example, and infinitely many thereafter. But how do we know that there are any such things? If numbers exist, they would surely be abstract, and there seems to be no way of explaining our knowledge of abstract objects, short of postulating some supernatural mode of apprehending them. Numbers in particular do not emit or reflect signals, they leave no traces, their behavior causes no phenomena from which their existence may be inferred. Although it is plausible that arithmetic is indispensable to empirical science, this has been disputed; and even if true, its significance for the truth of arithmetic and the existence of numbers is moot, as one might take an instrumentalist view of arithmetic in science.[1] Considerations of this sort have led some people to conclude that we do not have knowledge of numbers, and to seek to accommodate the prevalence and utility of arithmetic in one way or another within a nominalist framework.

[1] Hartry Field has argued against the claim that arithmetic is indispensable to science—*Science without Numbers* (Princeton: University Press, 1980). Some accept the claim but dispute its epistemological significance: Penelope Maddy, "Indispensability and Practice," this JOURNAL, LXXXIX, 6 (June 1992): 275-89; and Elliott Sober, "Mathematics and Indispensability," *Philosophical Review*, CII (1993): 35-57. Michael Resnik defends both the claim and its significance—*Mathematics as a Science of Patterns* (New York: Oxford, 1997). The indispensability criterion is stated in Hilary Putnam, "Mathematics without Foundations," in *Mathematics, Matter, and Method: Philosophical Papers*, Volume I (New York: Cambridge, 1967). Putnam attributes the criterion to W. V. Quine.

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Nominalism is the view that there are no abstract entities.[2] Nominalism about arithmetic is the view that either numbers are not abstract, or arithmetical sentences, correctly interpreted, do not entail the existence of numbers, or arithmetic is false. Of these three brands of nominalism, the first is not popular[3] and so I shall say no more about it. The task for the second brand is to find a plausible interpretation of arithmetic that maintains truth and avoids commitment to abstract entities.[4] A route favored recently is to go modal: interpret arithmetical statements as about possibilities or possible concrete objects. A common feature of these attempts is that under the chosen interpretation, the proposition expressed by an arithmetical sentence differs from anything we have in mind when using the sentence.[5] So this brand of nominalism is prima facie implausible, and it leaves open the question of the truth of the propositions we actually have in mind when using arithmetical sentences.

The third brand of nominalism about arithmetic is fictionalism. According to fictionalism, arithmetical sentences say just what we normally think they say, and our arithmetical theorems have the existential consequences we normally think they have; but arithmetical theorems with existential consequences are one and all false.[6] There are infinitely many primes? False! $2 + 1 = 3$? False! This is uncomfortable. Simple numerical equations are among the propositions that seem most obviously true. There are various emollients to alleviate discomfort: arithmetic does not have to be true to be useful, and we can specify a kind of correctness without appeal to truth:

[2] What this comes to is not clear, as the abstract-concrete distinction is hazy. But it is widely assumed that numbers would have to be abstract. For the difficulty of drawing the abstract-concrete distinction, see Bob Hale, *Abstract Objects* (Cambridge: Blackwell, 1987), chapter 3.

[3] The only one I know of who might qualify as a nominalist of this kind is Bishop Berkeley, who claimed that "numbers are nothing but names"—*Philosophical Commentaries* (New York: Garland, 1989), § 763, cf. § 803. It is not clear what he would have said about numbers beyond those for which a name has been uttered or written down.

[4] This is dubbed "the hermeneutic strategy" in John P. Burgess and Gideon Rosen, *A Subject with No Object* (New York: Oxford, 1997).

[5] For example, on one nominalist interpretation of arithmetic, an arithmetical sentence *S* is to be understood roughly as follows: there could be a model of the second-order Dedekind-Peano axioms for arithmetic and necessarily, for any model *M* of those axioms, *S* is true in *M*. Clearly, only someone with some knowledge of mathematical logic would even entertain a thought of this kind. For details, see Geoffrey Hellman, *Mathematics without Numbers: Towards a Modal-Structural Interpretation* (New York: Oxford, 1989). For an alternative nominalist view, see Charles Chihara, *Constructibility and Mathematical Existence* (New York: Oxford, 1990).

[6] See Field; and David Papineau, "Knowledge of Mathematical Objects," in A. D. Irvine, ed., *Physicalism in Mathematics* (Boston: Kluwer, 1989), pp. 154-81.

derivability from axioms. Moreover, there are various reconstructive strategies[7] for finding a nominalistic statement capturing "the grain of truth" in a theorem of arithmetic. But this satisfies only a lucky few. Surely, some misunderstanding is involved in disbelieving that 2 + 1 = 3, for example? Thus we may be trapped, stuck at one or another unsatisfactory position or oscillating between them.

The aim here is to show that this trap is avoidable, by arguing for the possibility of a naturalistic account of knowing numbers, without taking them to be concrete objects. I shall try to show that cognitive science already provides some of the resources for such an account. The numbers I have in mind are the finite cardinals: some we know by acquaintance and some by description. The paper takes these cases in order, and then something is said about knowing finite numbers more abstract than cardinals. But first I want to say what cardinal numbers are, and to present a general setting for the epistemology to follow.

I. WHAT ARE CARDINAL NUMBERS?

Cardinal numbers are properties. In my view, they are properties of sets, but they might also be properties of concept extensions, collections, pluralities, nonmereological aggregates, or some other kind of collective, provided that collectives of one or zero items are not excluded. I shall talk of sets for convenience, but I do not insist that only sets have cardinal number. As long as one is clear about which are the items in the collective to be numbered, one need not deploy a general collective concept in framing the question: What is the number of sheep in this flock? In fact, one can do without any collective concept, general or specific: How many sheep are here? Nonetheless, the answer to the question gives the cardinal size of the flock, and that is as much a property of the flock as its monetary value or the average age of its members.

But do any properties exist? Are properties and kinds not just a philosopher's fiction, or the result of a mistakenly literal reading of certain *façons de parler*? This view was famously expressed by W. V. Quine[8]:

> One may admit that there are red houses, roses and sunsets, but deny, except as a popular and misleading way of speaking, that they have anything in common. ...[T]he word 'red' or 'red object' is true of each of sundry individual entities which are red houses, red roses, red sunsets;

[7] Burgess and Rosen survey the reconstructive strategies.
[8] "On What There Is," in his *From A Logical Point of View* (Cambridge: Harvard, 1953), pp. 1-19.

> but there is not, in addition, any entity whatever, individual or otherwise, which is named by the word 'redness', nor, for that matter, by the word 'househood', 'rosehood', 'sunsethood' (*ibid.*, p. 10).

Plato[9] may have thought that for every general term there is a corresponding Form. Quine held that for no general term is there a corresponding property or kind. Between these two extremes lies the "mixed" view that for some but not all general terms does there exist a corresponding property or kind. Although there is no decisive argument for any of these views, I hold that the mixed view should be our default hypothesis. It is true that we rarely assert or deny the existence of a property or kind, as opposed to instances of a property or kind. But although talk of this kind is rare, judgments about the existence or nonexistence of properties and kinds are sometimes made in science. For example, Joseph Priestley thought that all combustible material contained phlogiston, a substance that is liberated from the material in combustion, with the dephlogisticated substance left as an ash or residue. Antoine Lavoisier thought that there was no such substance as phlogiston and no such property as being dephlogisticated.

Is this really a case of a difference about the existence of a kind or property, rather than about the existence of instances? Of course, they did disagree about the existence of instances. But there was more to it than that. On Lavoisier's view, nothing *could* contain phlogiston or be dephlogisticated. Compare this with the case of a transuranium element prior to evidence that there exist any atoms of that element. Before 1996, no sample of the element with atomic number 112 had been found and, as far as was known, there was none in existence. Yet it was held that there could be some, and in fact some was produced (by bombarding lead with zinc). This is the current situation for the element with atomic number 114.[10] Thus, scientists view the following as an epistemic possibility: there is no sample of element 114 but some could be produced. Although it is also believed that there is no sample of phlogiston, it is not also believed that some could be produced. What explains the difference? Surely, it is not that present technology for producing phlogiston lags behind present technology for producing superheavy elements, nor even that there is an insuperable technological barrier, such as an

[9] *The Republic*, G. M. A. Grube, trans. (Indianapolis: Hackett, 1992) 596a. *X corresponds to* general term *T* when *X* is designated by the singular term formed from *T* (for example, *T*-ness, or in Plato's writing, the *T* itself).

[10] By the time you read this, the claim may be out of date, but the philosophical point is unaffected.

unattainable energy requirement, to phlogiston production. The answer is that while there is such a kind as element with atomic number 114 and such a property as having a nucleus with 114 protons, there is no such kind as phlogiston and no such property as being phlogiston.

Are we being misled by popular ways of speaking here? Presumably, we have a naive linguistic disposition to generalize existentially on a noun phrase, and the idea that kinds and properties exist might be nothing but an unconscious conversion of this disposition into an ontological thesis, regardless of empirical justification. But in holding the mixed view (that some noun phrases for properties or kinds do designate while others do not), we clearly resist that disposition. On the mixed view, decisions about what kinds and properties are real is to be based on empirical findings. The extreme views, by contrast, issue from metaphysical prejudice, as they are unlikely to be accepted by anyone without a platonist or a nominalist bent.[11] For these reasons, I am going to take the mixed view as the default position and use it as a background premise. That of course leaves open the question whether there are such properties as cardinal numbers. The affirmative view is not confined to those with a general predisposition to platonism. John Locke,[12] for example, included number in the list of what he was prepared to call "real qualities." But this view needs vindicating. The challenge is to show how, if there were cardinal number properties, we could have knowledge of them. I shall try to meet that challenge in what follows by sketching a naturalistic account of such knowledge.

II. ACQUAINTANCE WITH CARDINAL NUMBERS

We say that we know John Doe, but for entities other than people, talk of knowing them is unusual and sometimes odd. But not always: on hearing the opening bars of a symphony, one might say 'I know that music'; on seeing the handwritten address on an envelope, one might say 'I know that handwriting'. Here one claims to recognize the

[11] At the time Quine wrote "On What There Is," he was not the (reluctant) Platonist that he later became. That article was first published in 1948, a year after the publication of "Steps towards a Constructive Nominalism," *Journal of Symbolic Logic*, XII (1947): 105-22, co-authored with Nelson Goodman. In that article they declare their belief in the nonexistence of abstracta to be a "philosophical intuition that cannot be justified by appeal to anything more fundamental" (p. 105).

[12] *An Essay Concerning Human Understanding* [1689] (New York: Oxford, 1975), Book II, Chapter VIII, § 17. One might argue that what Locke really had in mind here is not the number *n* but the property of having *n* atomic constituents ("solid parts"). But this reading of the passage is dubious and cannot easily be transferred to other passages in which Locke seems to say that number is a primary quality.

music or handwriting, and if the claim is true one must be acquainted with it.[13] In these cases, acquaintance entails that one has already heard performances or seen samples, and from this prior experience one has acquired a concept of the music or handwriting, and an ability to recognize the music or handwriting in other performances or samples, where this involves thinking of them under a relevant concept. Generalizing, I shall take it as sufficient for acquaintance with a property that one has experienced instances of it, and as a result one has acquired a concept of the property and an ability to recognize other instances and to discriminate them from noninstances (unless circumstances are unfavorable), where this involves applying the concept exclusively to the instances.[14]

What does recognitional ability consist in when the property concerned is a cardinal number? It is not enough to be able to tell whether a given set has the cardinality. By careful counting, we can tell whether a given set of stationary visible objects whose cardinality lies somewhere in the 200-to-300 range has cardinality 271. That would be mere detection rather than recognition. Recognition of n requires that we have some sense of the cardinal size n as distinct from its neighbors. For which cardinals do we have such a sense? There is evidence that we have an innately given magnitude representation of rough cardinal size, or *numerosity*, with a neural basis in the inferior parietal lobes.[15] The evidence comes from a variety of sources: experiments on healthy adults and children, clinical tests on brain-damaged patients, brain-imaging techniques, and studies on animals from parrots to primates.[16] The capacity for representing numerosities shares a couple of features with other innate rough-quantity senses, such as sense of duration, temperature, and pitch. These features are known as the *distance effect* and the *magnitude effect*. Crudely put, the distance effect is that the smaller the difference

[13] I take it as obvious that one would not be claiming to know that very performance of the symphony or those very ink marks on the envelope. What one claims to know is the type rather than any of its tokens.

[14] An anonymous referee points out that this account of property acquaintance sounds perfectly acceptable to an antirealist about properties who favors conceptualism. While this may be true, it is not in conflict with my aim at this point, which is to argue that, if there were cardinal-number properties, we could have knowledge of them. My reason for taking seriously talk of properties is given in the previous section.

[15] This is what Stanislas Dehaene refers to by the title phrase of his book: *The Number Sense* (New York: Oxford, 1997).

[16] See Brian Butterworth, *The Mathematical Brain* (London: Macmillan, 1999), chapters 3-6. This was published in the United States as *What Counts: How Every Brain Is Hardwired for Math* (New York: Free Press, 1999). See also Dehaene, chapters 1-3, 7, 8.

between two quantities (for a fixed mean), the harder it is to distinguish them; the magnitude effect is that the greater the mean of two quantities (for a fixed difference) the harder it is to distinguish them.[17] Difficulty of discrimination is measured by response times and error rates for tasks requiring comparison of the quantities. The pattern of response times and error rates is part of the data to be explained; the fact that this pattern is the signature of rough-quality sensing supports the view that in making number judgments of the kinds, tested subjects are sensing rough qualities.[18] Putting together the distance effect and the magnitude effect, it follows that in using our sense of numerosity to distinguish adjacent numbers, it will be easier for small numbers; and this is how it seems to be. There is evidence from studies with very young children and animals for a prelinguistic ability to discriminate cardinal numbers 1, 2, and 3. This ability may be provided by our numerosity sense, something that is predicted by a neural network model for this sense.[19] This prelinguistic ability is matched by an extremely fast and reliable ability in adults to sense the cardinal numbers of sets of 1, 2, and 3 visually presented items, known as *subitizing*.[20] So we can sense these cardinal numbers, and once we have concepts for these cardinal numbers, we can recognize instances of them and discriminate them from nonin-

[17] For precise statements of these features, see any textbook in psychophysics under "Fechner's Law" and "Weber's Law."

[18] Thus it is a mistake to think that the only data we have are data to the effect that we know facts of the form 'There are *n* so-and-sos' for small *n*. This, I hope, partly answers the question of an anonymous referee how such data justify the claim that we know numbers as objects. In fact, I do not want to claim that we know numbers as objects, and if some formulations suggest otherwise, please regard that as a slip. My claim is that a naturalistic account of knowledge of numbers is not impossible if we take cardinal numbers to be properties.

[19] Dehaene and Jean-Pierre Changeux, "Development of Elementary Numerical Abilities: A Neuronal Model," *Journal of Cognitive Neuroscience*, v (1993): 390-407. The hypothesis that the numerical abilities of animals and infants are based on a sense of numerosity is put forward by Dehaene in *The Number Sense* among others. This is disputed in Susan Carey, "Evolutionary and Ontogenetic Foundations of Arithmetic," *Mind and Language* (forthcoming). Carey's view is that we have the sense of numerosity but that performance of numerical tasks by animals and infants uses other cognitive resources.

[20] Subjects are asked to indicate the number of presented items flashed on a screen as quickly as they can. When response time is plotted as a function of number of items viewed, there is a discontinuity in slope: almost horizontal for small numbers (too fast for silent counting), steeper for larger numbers (matching increments in counting time). Error rates show the same pattern. The discontinuity persists under diverse display conditions. This is evidence for different processes of enumeration: a fast, error-free process for small numbers, that is, subitizing, and a slower, less reliable process for larger numbers. (The subitizing limit varies between 3 and 7, depending on type of experiment, visual display, statistical techniques used to calculate subitizing limit, and individual differences.)

stances (in particular, from instances of adjacent cardinal numbers), as we do in subitizing.

What kind of experiences would help us get concepts of cardinal numbers?[21] For small nonzero cardinal numbers, our experiences of counting would be a natural first call. But one has to be careful here. What does counting consist in? On some definitions, counting might prerequire concepts of cardinal numbers, in which case our acquisition of the concepts could not be the result of counting experience. Let us take counting to include the one-to-one assignment of words in a stable order with a perceptually given set of entities. That is, to be a competent counter a child must obey the stable-order principle and the one-one principle (allowing for slips): the counting words must be used in the same order in different counts and they must be assigned one-to-one with the items being enumerated.[22] A child who obeys the stable-order and one-one principles may still not know what counting is for. Even when a child knows that the correct answer to the question 'How many?' is given by the last word in the count, she may not understand the question as a request for the number of items counted.[23] Experimental results show that children come to realize that the last word in a correct count signifies the number of the objects counted only after they have acquired the stable-order principle and the one-one principle. That is, they learn the cardinal significance of counting only after learning correct counting procedure.[24] At about the same time, they come to know which cardinal number each word in their counting list denotes.[25] I assume that in

[21] It is possible that we have innate concepts of cardinals 1, 2, and 3. Experiments of Karen Wynn suggest that infants as young as five months make numerical predictions when numbers are limited to three—"Addition and Subtraction by Human Infants," *Nature*, CCCLVIII (1992): 749-50. Wynn's findings have been replicated many times under varying conditions. On one explanation of these findings, infants have genuine numerical beliefs, which have concepts for the cardinals 1, 2, and 3 as components. The explanation of these findings is under dispute, however.

[22] For counting principles, see Rochel Gelman and Charles Gallistel, *The Child's Understanding of Number* (Cambridge: Harvard, 1978), chapter 7.

[23] Children at this stage sometimes treat a number word as a variable designator, like a demonstrative, taking it to refer in a given count to the item to which it is assigned. See Karen Fuson, *Children's Counting and Concepts of Number* (New York: Springer, 1988); and Wynn, "Children's Understanding of Counting," *Cognition*, XXXVI (1990): 155-93.

[24] Wynn (*ibid.*) found that the children in her study acquire the cardinal principle at around the age of 3 ½ years, after they have learned correct counting procedure. This is contrary to the understanding-before-skill hypothesis of Gelman and Gallistel, but accords with the reverse hypothesis of Fuson.

[25] Wynn, "Children's Acquisition of the Number Words and the Counting System," *Cognitive Psychology*, XXIV (1992): 220-51. Wynn found that children take the

the process they acquire concepts for those cardinal numbers. How this happens is not clear. It is not improbable that part of the process involves mentally associating representations of initial words in the counting list with initial representations supplied by the sense of numerosity.[26] An additional possibility is that in learning to count we construct a category representation of sets of a given size, one for each set size from 1 to 3.[27] These might then serve as representations of those cardinal numbers and get mapped onto the initial numerosity representations. Our counting experience might also lead us to associate adding a further item to the set of those counted with proceeding from one cardinal number to the next one up in order of size. So our concept of cardinal 2, for example, would present it as the successor of 1 as well as the number between 1 and 3. Thus, experience of the counting procedure and of its use may feed into the acquisition of concepts for cardinal numbers 1, 2, and 3, and enable us to recognize new instances of each and discriminate them from instances of the neighboring cardinal numbers. In this way, we can have acquaintance with these numbers.

Do we have acquaintance with cardinal numbers greater than 3? My inclination is to think that we do. In our experience of counting, we repeatedly meet and notice numbers from 4 to 10, as finger counting typically has an important developmental role.[28] In so doing, we might sharpen our sense of numerosity to get senses of individual cardinal numbers beyond 3, just as repeatedly exercising

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number words in their list to stand for unique cardinalities before knowing which cardinality each one denotes. They may reach this stage through experience of the use of number words outside counting contexts, reaching the later stage only when they make an order-preserving association of number words in counting sequence with cardinals in order of size, perhaps by some general analogizing mechanism. See also Fuson, "Relationships Between Counting and Cardinality from Age 2 to Age 8" in Jacqueline Bideaud, Claire Meljac, and Jean-Paul Fischer, eds., *Pathways to Number* (Hillsdale, N.J.: Lawrence Erlbaum, 1992), pp. 127-49.

[26] There is clinical evidence for neural links between the sites in the left hemisphere of the numerosity sense and of the aural number word store (inferior parietal lobe and perisylvian cortex). See Butterworth, chapter 4; and Dehaene, *The Number Sense,* chapter 7. Presumably these links are formed when children learn the cardinal meanings of the number words.

[27] Perhaps by "abstracting" from the nature of the elements of *n*-membered sets and the order in which the members are given, as Georg Cantor suggested, we form a representation of the cardinal number *n*—*Contributions to the Founding of the Theory of Transfinite Numbers* I (1895), Philip E. B. Jourdain, trans. (New York: Dover 1955), §1. This is not to endorse Cantor's view that these representations are the cardinal numbers.

[28] See Fuson, "Relationships between Counting and Cardinality from Age 2 to Age 8." For the neuropsychological importance of finger counting, see also Butterworth, chapter 5.

one's visual capacity for discriminating shades of red or types of snow sharpens our sense of those different properties or kinds. From counting experience, we get a sharpened sense of the cardinal size 4. Our sense of 4 may not be quite as clear and strong as our sense of 3, but the difference may not be great. Similarly, we may develop a sharpened sense of subsequent numbers, each almost as clear and strong as our sense of its predecessor. It is possible that there is a relatively sudden drop in the sharpness of the sense of numerosity thus developed, perhaps after 10, but I am not aware of evidence for it. If the contrary is true, there would be no natural cut-off point separating those cardinals we can recognize and those we can only detect; consequently, acquaintance with cardinals would have to be a matter of degree. But there is no ground for worry here, as that is how it is anyway for acquaintance with material individuals.

III. KNOWLEDGE OF CARDINAL NUMBERS BY DESCRIPTION

Small numbers beyond 3 may be known by acquaintance. But even if they are not, they are known by description through familiarity with small number addition. The cardinal number 5, for example, may be known as the size of a set composed of a triple and a pair (without repetition or overlap). Clearly, all the numbers from 4 to 10 can be easily and briefly described this way without assuming knowledge of cardinal numbers beyond 3. Early addition exercises performed using counting techniques give us knowledge of all two-addend cardinal sums with totals up to and including 10.[29] Thus, for each number from 4 up to 10 there are several descriptions of this sort (sum of cardinal x and cardinal y) by which we know it; and we have more of these descriptions for larger numbers, so that our richer descriptive knowledge of larger numbers makes up for our weaker direct sense of them.[30]

This kind of knowledge takes us up to 10, perhaps to 20, perhaps even to 100. What about knowledge of much larger numbers? There are many possibilities, but probably just a few basic kinds of description that most of us use.[31] As multidigit numerals (in the decimal

[29] For a developmental account, see Fuson, "Relationships between Counting and Cardinality from Age 2 to Age 8."

[30] The cardinal number 0 may (must?) be known by description—for example, the number of wild tigers in Scotland, the result of taking away two things from a set of two. Perhaps one does not think of '0' as naming a cardinal number (rather than an indicator of absence) ahead of accepting that there is an empty set. I do not know of empirical work on this question.

[31] I have in mind people who are not calculating prodigies but have a level of schooling in arithmetic that is standard in North America and Europe. I am not suggesting that this standard is very good or better than standards elsewhere.

place system) play such a big role in numerical thinking, it is natural to look to them for clues. An obvious thought is that a multidigit numeral is short for the corresponding polynomial description: the cardinal 271 is the cardinal of a set composed of 2 hundreds, 7 tens, and 1 one. This seems plausible and can clearly be extended using greater powers of ten. But we would need to say how we know of those powers of ten. This is easy for the case of a hundred, which is thought of as a set of 10 tens. But for higher powers the question arises whether our knowledge of the cardinal number really depends on its description in terms of powers. Do we think of the cardinal 2,710,003 as the cardinal of a set composed of 2 tens-to-the-power-of-six, 7 tens-to-the-power-of-five, and so on? Do we not first think of a ten-to-the-power-of-six as the number of things denoted by a '1' followed by six '0's? The actual story may be quite complicated. But whatever it is, we clearly *could* come to know cardinal numbers beyond 10 under the polynomial description, using a recursive definition for powers of ten: a ten-to-the-power-of-two is 10 tens; a ten-to-the-power-of-n+1 is 10 tens-to-the-power-of-n.

On the suggestion just made, we know larger numbers under their descriptions as polynomials, with multidigit numerals serving as the descriptive expressions. There is an alternative, in which the multidigit numerals play a more intimate part in our knowledge of numbers: the numerals are not merely the carriers of the description but are part of the descriptive content. How is this going to avoid circularity? If we think of the number 27176 as the number denoted by '27176', that cannot be the whole story. Through our knowledge of an algorithm for determining the numeral immediately preceding any given multidigit numeral and our possession of a concept for the cardinal successor function, we can amplify as follows: the number denoted by '27176' is the cardinal successor of the number denoted by the numeral predecessor of '27176'. Of course, this works only for someone who has knowledge of the number denoted by '27175'. But the same kind of description is available, using the following general schema: the number denoted by a multidigit numeral μ is the cardinal successor of the number denoted by the numeral predecessor of μ. Repeated application brings us back to single-digit numerals whose denotation we know by direct association with cardinals with which we are acquainted. In this way, circularity is replaced by recursiveness. I am not suggesting that we actually run back in thought from a multidigit numeral '27176' to single digits in thinking of the number it denotes. We just think of the number as the number denoted by '27176'; but our understanding of this description (in terms of numeral predecessor and cardinal successor functions) suffices to fix its reference. That is an alternative possibility.[32]

Probably the actual way in which we think of multidigit numbers is more complicated than either of the two suggestions made here, perhaps involving a mixture: the polynomial description for numbers of two to four digits and the apparently circular description in terms of numerals for larger numbers. Even this is liable to be an oversimplification, as it ignores how natural-language expressions for larger numbers affect how we think of them. On top of this there will be variations among people and over time. But my aim here is not to give "the correct account" of our knowledge of numbers by description. It is merely to show that such an account is possible without positing powers that could not be countenanced in cognitive science.

IV. KNOWING THE OBJECTS OF PURE NUMBER THEORY

Even if my case for the possibility of a naturalistic account of knowing cardinals is accepted, one might protest that we still lack a parallel case for the objects of number theory. Those objects are not set sizes; they are, if anything, positions in the natural number structure.[33] I am sympathetic to this view. Pure mathematicians at work in number theory are not concerned with finite cardinals as opposed to finite ordinals and the myriad other systems that exemplify this structure; they are concerned with the structure itself. Hence numerals and variables in the context of pure number theory are used to refer to (or range over) positions in the structure. But how could anyone have knowledge of such highly abstract objects, positions in a structure?

Although it lies beyond the scope of this paper to answer this question in any detail, I think an outline is possible. First, we would have to have knowledge of the structure itself. Assuming that we cannot have acquaintance with more than a finite portion of an infinite structure, knowledge of the structure must be knowledge by description, a description whose components are inferred from other knowledge. One possibility is that we derive a description of the natural-number structure from our knowledge of the system of nu-

[32] Neither of these accounts is intended to show how we might know all finite cardinals. That would be impossible, as there is a finite upper bound on the number of possible neural states, hence on the number of representations (sensory or descriptive) that the mind can grasp.

[33] More than one structure may be described as "the" natural-number structure, depending on the relations considered. For simplicity, we can take the sole relation to be the less-than relation. For structuralist views of the ontology of pure mathematics, see Stewart Shapiro, *Philosophy of Mathematics* (New York: Oxford, 1997), part II; and Resnik, part III.

merals in order of precedence, which is one of the systems having that structure. I assume that we are acquainted with some numerals (types, not tokens) and that we know about the system as a whole through our knowledge of algorithms for constructing the successor and predecessor of a given numeral, and for addition on the numerals. Writing $|\kappa|$ for the number of predecessors of numeral κ,[34] $\mu + \kappa$ is the numeral constructed by $|\kappa|$ applications of the successor operation starting from numeral μ. Precedence is understood in the familiar way: $\mu < \nu$ if and only if for some κ other than '0', $\mu + \kappa = \nu$. Through standard arguments we know that the order of precedence is a strict total ordering having just one element that is not a successor and no element that is not a predecessor. In fact, it is a well ordering, because from any numeral μ one can reach '0' by $|\mu|$ applications of the predecessor operation, and so there is no unending sequence of ever earlier numerals; hence every nonempty set of numerals has an earliest member. With this knowledge of the structure, as a well-ordering with one initial[35] position and no terminal position, we can know a position as the n^{th} position of the structure. For any numeral not too long to be grasped, we can know and name the position that it occupies and all preceding positions.

That is my outline. I concede that we are still some way off from being able to fill in the cognitive details of a naturalistic account of our knowledge of the natural-number structure, hence of its positions. But I am aware of no reason to think that future developments in the empirical study of our cognitive capacities will not one day provide the resources for a full account along the lines of the sketch just given.

V. CONCLUSION

Among philosophical attitudes to arithmetic prevalent today is a kind of skepticism: the theorems of arithmetic are not true, unless they are interpreted in one or another strange way. One of the main motivations for this attitude is the conviction that there could not be a naturalistic mode of knowing numbers. I have tried here to rebut that conviction. Drawing on recent work in cognitive science, I have sketched ways in which we might have knowledge of cardinal numbers. I have not canvassed all the possibilities, and it may be that none of those presented here are right. My claim is only that they, or something similar, could be right; so, I conclude, we should not

[34] I assume that '0' is the first numeral. So $|\kappa|$ is the cardinal denotation of κ.

[35] This terminology is slightly misleading. An *initial* position is one that is not the successor of another position. So the position of any limit ordinal is initial (as is the position of 0).

dismiss the very possibility of explaining knowledge of cardinal numbers within a scientific framework.

What has prevented people from seeing the kind of possibilities outlined here? There are probably several obstacles. Here are my suggestions. One has to do with the word 'object'. If cardinal numbers exist, they are objects, we say. That dictum is a truism if the word 'object' is used to mean 'entity'. But cardinal numbers, being properties of sets, are very unlike prototypical objects, which are bounded, perceptible, spatiotemporal continuants. Moreover, the word 'object' is often used contrastively with 'property'. So the dictum may obscure the fact that cardinal numbers are properties and incline one to think that, if there were such things as cardinal numbers, they would have to be something like disembodied billiard balls. A second obstacle may be a nominalist reluctance to accept the existence of properties. A third obstacle may be that the empirical study of cognitive capacities for number knowledge is relatively young—though it is now a vibrant research subject. Finally, philosophers of mathematics have shown a near total disregard for the empirical findings, despite a willingness to pronounce on cognitive matters.[36] I hope that this paper shows the merit of abandoning that attitude.

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[36] A notable case is Philip Kitcher's claim that children come to learn the meaning of 'number' and to accept basic truths of arithmetic "by engaging in *activities* of collecting and segregating"—*The Nature of Mathematical Knowledge* (New York: Oxford, 1984), chapter 6. Kitcher gives no reference to empirical work on children's cognitive development, and his view is not borne out by the findings earlier published in Gelman and Gallistel.

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[Footnotes]

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[1] **Mathematics and Indispensability**
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The Philosophical Review, Vol. 102, No. 1. (Jan., 1993), pp. 35-57.
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[11] **Steps Toward a Constructive Nominalism**
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