Unlocking Limits

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Abstract

In a series of recent papers we have developed what we call the DEKI account of scientific representation, according to which models represent their targets via keys. These keys provide a systematic way to move from model-features to features to be imputed to their targets. We show how keys allow for accurate representation in the presence of idealisation, and further illustrate how investigating them provides novel ways to approach certain currently debated questions in the philosophy of science. To add specificity, we offer a detailed analysis of a kind of key that that is crucial in many parts of physics, namely what we call *limit keys*. These keys exploit the fact that the features exemplified by these models are limits of the features of the target.

Keywords: Scientific modelling, Representation, Limits, Keys, DEKI.

1. Introduction

Many scientific models are representations of a target system, a selected part or aspect of the world. To understand how these models work we have to understand how representation works. In our (2016, 2018) we formulate the DEKI account of scientific representation which assigns a central role to what we call a *key*: a systematic way for moving from model-features to features to be imputed to the models' targets.[1] To the extent that their targets have those features, the models in question are accurate representations.

So far we have discussed the account at a relatively high level of abstraction and said rather little about how keys work. But to understand how a model represents it is crucial to know the details of the key that accompanies it. The aim of this paper is to start filling this lacuna in the DEKI account by characterising a typical kind of key associated with many models in physics, namely what we call *limit keys*. This kind of key exploits the fact that the features of models are

[1] For a discussion of alternative accounts of representation see our 2017, 2020.

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the results of taking certain features of the target system to a limit. Appropriately understood, these keys allow for models that radically diverge from their targets—in the sense that they are highly idealised—to nevertheless represent them accurately. As such, by making these keys explicit, the epistemic role of certain kinds of idealisation is clarified. However, as we will see, a limit key can only be invoked under particular conditions. Specifying these conditions forces us to pay careful attention to certain choices scientists make in the construction of their models, and doing so sheds a new light on certain controversies about models. Thus, this paper's contribution is threefold. First, it develops the DEKI account of scientific representation by adding an analysis of limit keys. Second, it illuminates a certain area of scientific practice by scrutinising the epistemic function of taking target-features to a limit in a model. Third, it demonstrates how such models can be accurate despite being idealised, thereby contributing to our understanding of the epistemic value of idealisation.[2]

We proceed as follows. In Section 2 we briefly recapitulate the DEKI account of representation. In Section 3 we introduce limit keys. Section 4 illustrates how, and under which conditions, they work via some simple examples. Section 5 discusses the methodological assumptions that underpin the use of limit keys 'in the wild', where the relevant features that have been taken to a limit, and the nature of these limits themselves, are assumed (as part of scientific practice, rather than rigorously proven) to be relatively well-behaved. Section 6 concludes.

Before we begin, it's worth commenting on how the use of limit keys to underpin cases of scientific representation contributes to our broader philosophical account of scientific modelling. We have argued elsewhere (Frigg 2010a, 2010b; Frigg and Nguyen 2016) that scientific models should be thought of as akin to works of fiction. Now, it's important to note that this claim concerns the *ontological* status of scientific models. As such, it is only a *part* of a complete philosophical account of model-based science. The fiction view of models tells us what models are, but not how they function representationally. The DEKI account of scientific representation is thus designed to supplement the fiction view by providing an account of how a scientific model, thought of as a work of fiction, might represent a target system. For our current purposes, the fictional nature of the models in question is left in the background, since we focus on the nature of a particular kind of key.[3]

2. DEKI

The DEKI account of scientific representation provides a general framework for thinking about the representational relationship between models and their targets. The framework specifies four conditions that must be met for a scientific model M to represent a target system T so that reasoning about the former can

[2] Our account thus avoids regarding idealised models as falsities, or misrepresentations. This comes at the costs of rejecting the notion that models have to be interpreted literally. For a discussion of this point see our 2019 and Nguyen 2019.

[3] But see our 2016 for a discussion of the interplay between DEKI and the fiction view of models more generally.

generate hypotheses about the latter. The conditions, which also give that account its name, are denotation, exemplification, keying up, and imputation.

The first condition is that *M denotes T*. Denotation is a two-place relation. A name denotes its bearer; a map denotes its territory; a portrait denotes its subject; and a model denotes its target. Denotation is necessary but insufficient for scientific representation. It's necessary because it establishes the bare sense in which *M* is about *T*. It's insufficient because it doesn't account for how we can reason about target systems via investigating their models, which is what Swoyer (1991) calls 'surrogative reasoning'. DEKI's other three conditions are designed to explain this.

The second condition is that models *exemplify* certain features.[4] Exemplification is instantiation plus reference: something exemplifies a feature if it at once possesses that feature and refers to it. This can be illustrated with Goodman's (1976: 52-56) example of a tailor's book of fabrics. The swatches both instantiate the particular kind of cloth they are—e.g. herringbone or pin-stripe—and also refer to these cloth-properties themselves.

Now, whilst scientific models may exemplify certain features, these features needn't be carried over to their target directly. A piece of litmus paper dipped into an acidic solution exemplifies redness, but it doesn't represent the solution as being red. Rather, the litmus paper—understood as a representation—comes with a *key* which systematically relates colours to pH values. Similarly, whilst a map exemplifies a certain distance between, say, the marks that are labelled 'Newcastle' and 'London', this distance isn't carried over directly to the cities themselves: rather the map comes with a key specifying a scale with which to systematically relate map-distances to the actual distances that the map represents. The DEKI account insists, and that's the third condition, that models function like litmus paper or maps in that they come with a key that associates model-features with target-features. In general terms, a key is a mapping which takes as arguments the exemplified features $P_1, ..., P_n$ of *M* and delivers as values some (possibly, but not necessarily, identical) features $Q_1, ..., Q_m$.[5]

The final condition is that the model user *imputes* at least one of $Q_1, ..., Q_m$ to *T*. If *T* has the feature imputed, then the representation is accurate in that respect. If it doesn't, then *M* still represents *T* as having such a feature; it's just a *mis*representation in that respect.

Tying these conditions together delivers:

DEKI *M* represents *T* iff

1. *M* denotes *T*;
2. *M* exemplifies features $P_1, ..., P_n$;
3. *M* comes with a key *K* which associates exemplified features $P_1, ..., P_n$ with features $Q_1, ..., Q_m$; and
4. a model user imputes at least one of $Q_1, ..., Q_m$ to *T*.

[4] We place no restrictions on what counts as a feature. In the current context, (one-place) properties, *n*-place relations, functions, solutions to equations of motion, and structural relationships, among others, count as features.

[5] We are not claiming that there is an easy way to dissociate different model-features, nor that the key is insensitive to relationships between them. This is just a schematic rendering of how keys work, additional constraints may be required.

DEKI provides a general framework in which to think about the relationship between models and their targets, and the framework needs to be filled in in particular cases. In order to understand a particular instance, or style, of scientific representation, the ways in which the conditions are met need to be further explicated. Our concern in this paper is the third condition. What associations between model-features and features to be imputed to the target are there, and how does a key encode them? Our goal here is to illustrate how the account works, and to illuminate a particular kind of reasoning, namely where the key in question exploits the notion of a limit. As we discuss below, by analysing this kind of reasoning in terms of DEKI, we also gain additional understanding of the role of (at least one kind of) idealisation in science.

3. Limit Keys

Many models exemplify 'extremal' features: model-planes are frictionless, model-gases have an infinity of molecules, and model-planets are perfect geometrical spheres. What do models exemplifying such features tell us about target systems that don't, and never will, have such features? The core idea that we develop here is that (at least some) models of this kind should be interpreted as being equipped with a limit key: a key that exploits the fact that the model-features can be understood as resulting from taking certain features of the target to a limit.

To give a definition of limit keys and analyse them, we must first introduce limits. We restrict our attention to two cases: number sequences and function sequences. A *number sequence* is a list of numbers linked by a rule. The list is usually indexed by an index α and the rule is given by an operation. As an example, consider the sequence $1/\alpha$ for $\alpha = 1, 2, 3, \ldots$. We follow an often-used convention and write such sequence as f_α. In our example we have $f_\alpha = 1/\alpha$. Although intuitive, nothing depends on the index being a natural number (in the next section we will see an example where α is a real number).

We can now ask how f_α behaves if α tends toward infinity. That is, we can consider the *limit* of f_α for $\alpha \to \infty$, where the symbol '∞' denotes infinity. If that limit exists and has value L, we write $\lim_{\alpha\to\infty} f_\alpha = L$. The question now is how a limit can be defined precisely and under what circumstances it exists. The standard definition of a limit is couched in terms of positive real numbers ϵ (where 'positive' means $\epsilon > 0$). These numbers can be arbitrarily small, but never equal to 0. Then, the limit L of the sequence f_α for $\alpha \to \infty$ is defined as follows:[6]

(1) $\lim_{\alpha\to\infty} f_\alpha = L$ iff $\forall\epsilon > 0\ \exists\alpha'$ such that $\forall\alpha$: if $\alpha > \alpha'$, then $|f_\alpha - L| < \epsilon$.

Intuitively this means that we can keep f_α as close to L as we like by making α sufficiently large. Limits with $\alpha \to \infty$ are also referred to as infinite limits. If it is not possible to keep f_α as close to L as we like by making α sufficiently large, then the infinite limit does not exist. If the limit exists, we say that the sequence f_α converges toward L. Consider again the previous example of $f_\alpha = 1/\alpha$. We can now take the limit of this sequence for $\alpha \to \infty$ and it is obvious that $\lim_{\alpha\to\infty} 1/\alpha = 0$.

Infinite limits can be taken irrespective of whether α is a natural number or a real number. When we look at cases where α is a real number, we can also ask

[6] This, and the below definition of a finite limit, are standardly stated in books on calculus. See, e.g., Spivak 2006: Chapter 5.

how the sequence behaves when α tends toward a particular (finite) value a. For instance, we can ask how f_α behaves when α tends toward zero, or toward five. The standard definition of such a limit is couched in terms of two positive real numbers, ϵ and δ (where, as previously, 'positive' means that both $\epsilon > 0$ and $\delta > 0$). The definition then says:

(2) $\lim_{\alpha\to a} f_\alpha = L$ iff $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall \alpha$: if $0 < |\alpha - a| < \delta$, then $|f_\alpha - L| < \epsilon$.

Intuitively this means that we can keep f_α as close to L as we like by keeping α close to a. If this is not possible, then the limit does not exist.

It's crucial not to conflate the limit of a sequence with the value of the sequence at the limit: L and f_a are not the same mathematical objects. To see this, consider the case where $\alpha \to a$. Since the definition of the limit requires $0 < |\alpha - a| < \delta$ (that is, the limit requires that $|\alpha - a|$ has to be strictly greater than 0), α will never be equal to a in taking the limit. So the limit L reflects how f_α behaves when α comes arbitrarily close *a without reaching it*. It does *not* reflect the value of f_α if $\alpha = a$. The same holds for infinite limits: because α tends towards ∞ without ever reaching it, L is not the same as f_∞. To express this difference clearly, we call L the *limit value* and refer to f_α (or f_∞) as the value at the limit.[7]

That two values are conceptually distinct does not mean that their numerical values must be different. If both the limit value and the value at the limit exist and are equal, then the limit is a *regular limit*; if they are different it's a *singular limit* (Butterfield 2011: 1077).[8]

We will see examples of both cases later. Before discussing examples, we can now say what a limit key is. Let the target system have a feature of interest corresponding to some value in the sequence f_α. To study the target, we construct a model in which the parameter α assumes the extremal value. Let us begin with a finite value a. This means that the feature exemplified by the model is f_a. Now assume (i) that the limit L of f_α exists for $\alpha \to a$; (ii) that the value f_a at the limit exists; and (iii) that the limit is regular (i.e. that $L = f_a$). Under these assumptions it follows that for all ϵ there exist a δ such that for all α, if $|\alpha - a| < \delta$, then $|f_\alpha - f_a| < \epsilon$. This can be exploited. If we consider a limit $\alpha \to a$, the model user can infer that as long as α in the target is not more than δ away from a in the model, the value of f_α in the target is no more than ϵ away from f_a in the

[7] In cases where the extremal value is ∞, the below discussion regarding the value at the limit requires that we specify what this value is. In the case of number sequences we can follow Butterfield (2011: 1075) and consider the sequences as containing elements from $\mathbb{N} \cup \{\infty\}$ (or $\mathbb{R} \cup \{\infty\}$), where '$\mathbb{N}$' denotes the natural numbers and '$\mathbb{R}$' denotes the real numbers. This is standard practice in the physics literature where the idea of a 'natural infinite system' corresponding to a system at an infinite limit is often invoked; see, for example, Ruelle's discussion of phase transitions as only occurring in systems that are 'idealized to be actually infinite' (2004: 2).

[8] Although note that Butterfield recommends caution with respect to the use of the term 'singular limit', given the variety of meanings one finds in the literature (see Butterfield 2011: 1068). It's worth noting here that Butterfield uses the phrase 'non-singular' limit to refer to both cases where the limit exists, and is equal to the value at the limit, and cases where the limit exists and there is no obvious value at the limit. Given our current purposes (where we're investigating models which are 'at the limit' so to speak), our use of 'regular limit' is restricted to the first kind of 'non-singular' limit.

model. Or, more colloquially, if the parameter α in the target is close to the model value, then the feature f_α in the target is close to f_a in the model. In this way knowing the model feature gives information about the target feature. If a model user employs knowledge of limits in this way to infer from a model-feature to target-feature she uses a *limit key*. Such a key works by taking the exemplified feature in the model, f_a, and converting it into a logically weaker property: having a feature in the interval $(f_a - \epsilon, f_a + \epsilon)$. It is this weaker feature that is imputed to the target system. In terms of the symbolic notation introduced in the last section, f_a is P and Q is having a feature in the interval $(f_a - \epsilon, f_a + \epsilon)$.[9]

The argument is *mutatis mutandis* the same if we consider an infinite value. In this case the feature exemplified by the model is f_∞. Assume that the limit for $\alpha \to \infty$ is regular. Then the model user can infer that if α in the target is larger than a threshold α', then the value of f_α in the target is no more than ϵ away from f_∞ in the model.

We can now turn to *function sequences*. The difference between number sequences and function sequences is that a function sequence is not a sequence of numbers but a sequence of functions $f_\alpha(x)$. The functions can be of any kind, but to keep things simple we consider real valued functions: $f_\alpha : \mathbb{R} \to \mathbb{R}$, where, as before, '$\mathbb{R}$' denotes the real numbers. An example of such a sequence is $f_\alpha(x) = x^{-\alpha}$. A function sequence can converge toward a limit function in different ways. One of the simplest is *pointwise convergence*: the function sequence $f_\alpha(x)$ converges pointwise toward the function $L(x)$ iff for every $x \in \mathbb{R}$ the value of $f_\alpha(x)$ converges to $L(x)$. If this is the case, we write $\lim_{\alpha \to a} f_\alpha(x) = L(x)$, and *mutatis mutandis* for $\alpha \to \infty$. We call $L(x)$ the *limit function* and $f_a(x)$ the *function at the limit*. As before, the limit function and the function at the limit can, but need not, be the same. If they both exist and are identical, then the limit is regular; if not, then it's singular.

Function sequence limits can be used to reason with the model about the target in the same way as number sequence limits. If the limit is regular it follows that for all x and for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all α, where $|\alpha - a| < \delta$, we have $|f_\alpha(x) - f_a(x)| < \epsilon$ (and, again, *mutatis mutandis* for $\alpha \to \infty$). This means that as long as (for each value of x) α in the target is not more than δ away from a in the model, the function $f_\alpha(x)$ in the target is no more than ϵ away from $f_a(x)$ in the model.[10] The limit key works by taking the exemplified feature of interest in the model, $f_a(x)$, and converting it into a logically weaker feature of interest, namely that the target's feature is somewhere in the interval $(f_a(x) - \epsilon, f_a(x) + \epsilon)$ for all x, which is imputed to the target system.

4. Toy Examples: Stairs and Slopes

Let's see this kind of reasoning in action with two toy examples: one where it works and one where it breaks down. In order to understand a method, it's often illustrative to see where it fails. So we start with an example, based on a number

[9] We drop the subscripts on the P and Q from here on for ease of notation since we're only dealing with a single exemplified model-feature and connecting it to a single feature to be imputed to the target.
[10] Since we're using the notion of pointwise convergence, the values of δ (and ϵ) can vary across different values of x.

sequence with a singular limit, where the limit reasoning fails. We then turn to an example where it works via a function sequence with a regular limit.

Assume that your target system is a set of stairs that you want to carpet. To buy the right amount of carpet you need to know the stairs' total length. The staircase in which the stairs are located has the shape of a right-angled triangle with both sides having unit length, and with the stairs sitting on the hypotenuse. Further suppose that there are a large number of stairs in the staircase and you somehow cannot work out their total length. You therefore resort to a model.

Let $\alpha = 1, 2, \ldots$ be the index of a number sequence. You start with a staircase with two stairs and every time you progress to the next index you double the number of steps in the staircase: for $\alpha = 1$ the staircase has two steps, for $\alpha = 2$ four steps, for $\alpha = 3$ eight steps, and so on. This is illustrated by the three images to the left in Figure 1. In general, for staircases in our sequence, the staircase with index α has 2^{α} steps. The dependant feature of interest, f_{α}, is the length of the stairs with index α; that is, f_{α} is the length of the set of stairs with 2^{α} steps. The number of steps seems so large to you that your model is a fictional scenario in which the stairs consist of an infinite number of steps. But a staircase with an infinite number of steps is a line, and so this idealisation results in a model, as shown by the 'staircase' to the right in Figure 1, where the length of the stairs is the length of the hypotenuse of a right-angled triangle whose other sides are of unitary length: $f_{\infty} = f_{\text{model}} = \sqrt{2}$.

You of course know that the number of steps is not infinite, but you think that this is not a problem because you can use a limit key. The number of steps is large, and you think that it is in fact large enough for the length of the model-stairs to be close enough to the length of the real stairs for all practical purposes, in particular to buy the right amount of carpet.

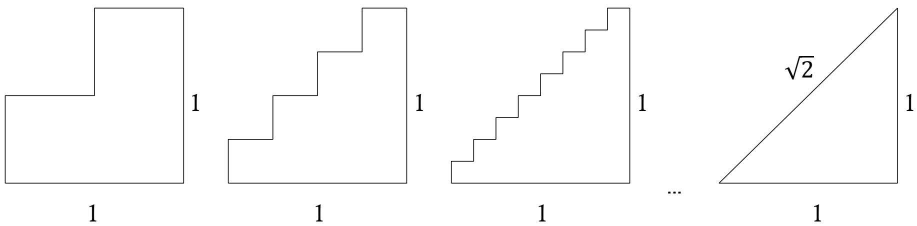

Figure 1: A sequence of staircases, with the value at the limit

This is mistake. Looking at Figure 1, it's easy to see that the total length of the stairs is two *irrespective* of the number of stairs: $f_{\alpha} = 2$ for all $\alpha = 1, 2, \ldots$. Hence, trivially, $\lim_{\alpha \to \infty} f_{\alpha} = 2$. So $L \neq f_{\text{model}}$. This shows that the limit is singular and we're now in a position to see how reasoning with a limit key breaks down (we're using definition (1) since we're dealing with an infinite limit). From $\lim_{\alpha \to \infty} f_{\alpha} = 2$ we know that for every $\epsilon > 0$ there is an α' such that: for all α, if $\alpha > \alpha'$, then $|f_{\alpha} - 2| < \epsilon$. But applying the limit key would amount to mistakenly assuming that for all $\epsilon > 0$ there is an α' such that for all α, if $\alpha > \alpha'$, then $|f_{\alpha} - \sqrt{2}| < \epsilon$. This is false. In fact, for any $\epsilon < 2 - \sqrt{2}$ there is no α' such that for all α, if $\alpha > \alpha'$, then $|f_{\alpha} - \sqrt{2}| < \epsilon$. So no matter how many stairs there are, the length of the stairs doesn't come close to the length of the hypotenuse, not even in the limit for the number of stairs toward infinity! This is why the limit key doesn't work here, and you would buy the wrong length of carpet if you were to reason in this way. So by using a limit key in a case where the limit in question is singular, the model yields wrong results.

Our second example works with a function sequence and provides an illustration of a case where limit keys work. Suppose your target system is a ski-jumper and you want to know how her position on the slope changes through time. To this end you construct a model, which is a fictional scenario consisting of a rectangular object sliding down a perfect geometrical plane with an inclination of θ. The materials of the object and the plane are such that there is no friction between them, and the only force acting on the object is the linear gravitational force $\vec{F} = mg$, where g is the gravitational constant on the earth's surface. With some simple trigonometry we can calculate the magnitude of the component of the force acting on the object parallel to the surface of the plane: $f^{\parallel}_{model} = mg \sin(\theta)$, as displayed in Figure 2.

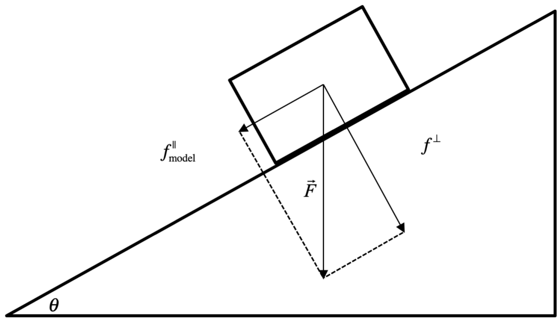

Figure 2: Ski-jumper model

Using Newton's equation, and without loss of generality setting the original position and initial velocity to zero, delivers the following position function along the slope for the object:

(3) $x_{model}(t) = 1/2t^2g \sin(\theta)$.

This function is an exemplified feature of the model, and in the idiom of DEKI it is *P*.

We know perfectly well that the real slope isn't a frictionless perfect plane, and that there are forces other than gravity acting on the skier such as air resistance and the Coriolis force. Given this, what does the model tell us about the real-world skier's position? To answer this question we need a key. In keeping with the spirit of our above discussion, we understand the model as a limiting case of the real situation and aim to construct a limit key.

To make a start, let us assume that the only force acting on the skier not taken into account in the model is friction, and that friction is linear. This is a strong assumption and we come back to it later; let's run with it for now to see how the reasoning works. The magnitude of the friction force acting on the skier then is proportional to the magnitude of the force perpendicular to the plane, $f^{\perp} = mg \cos(\theta)$, where the proportionality constant is the friction coefficient μ. Then, the actual force acting on the skier parallel to the slope is given by $f^{\parallel} = mg \sin(\theta) - mg\mu \cos(\theta)$. This means that the actual position function of the skier is:

(4) $x_{\text{friction}}(t) = 1/2t^2g(\sin(\theta) - \mu\cos(\theta))$.

Now regard μ as a freely varying parameter and notice the following relationship between $x_{\text{model}}(t)$ and $x_{\text{friction}}(t)$:

(5) $\lim_{\mu \to 0} x_{friction}(t) = x_{\text{model}}(t)$.

To see this, and to connect it to our above definition of a limit, it suffices to notice that the relevant δ for each ϵ is given by:

(6) $\delta = 2\epsilon/\cos(\theta)t^2$.

It's then easy to see that the condition in definition (2) is satisfied (for all values of *t*) and that the limit function is equal to the function at the limit. Hence the limit is regular. This allows us to use a limit key: for all times *t* and for any $\epsilon > 0$, it is the case that as long as $\mu < \delta$, it's guaranteed that $|x_{\text{friction}}(t) - x_{\text{model}}(t)| < \epsilon$. In words: as long as the friction coefficient in the actual system is less than δ, the position function in the model will differ from the actual position function by less than ϵ.

In the terminology of DEKI, the feature exemplified by the model, *P*, is $x_{\text{model}}(t) = 1/2t^2g\sin(\theta)$. The feature *Q* is: the position of the skier in the target is in the interval $(x_{\text{model}}(t) - \epsilon, x_{\text{model}}(t) + \epsilon)$ at all times *t*, where ϵ depends on the lower bound the model user can set on the value of μ in the target. The key then acts to connect feature *P* to feature *Q*. We can think about the key as a mapping from the exemplified features to the features to be imputed to the target. So, $K(P) = Q$. The value of the key, i.e. the feature *Q*, is then imputed to the target. Interpreted in this way, the model is an accurate representation (because the position function of the skier does actually fall within the bound imputed).

It's important to note that this doesn't rely on the idea that the friction acting on the skier is in any sense negligible or makes no difference to her movement. The exact same reasoning can be applied to all skiers irrespective of what the level of friction in the target is. Even if friction plays a significant role in the target system, Equation (6) can be used to say how the real skier moves in exactly the same way in which it is used in situations in which friction is small. We can use the frictionless model to impute *Q* as above, the only difference being that the interval defining *Q* is wider. And this will still result in the model being an accurate (albeit logically weak) representation.[11]

5. Limits in the Wild

Let us now return to our assumption that that the only force acting on the skier not taken into account in the model is friction, and that friction is linear. This assumption allowed us to specify the ϵ and the δ explicitly and prove that the limit exists. We made this assumption to illustrate how limiting reasoning works. It is, unfortunately, unrealistic in two ways. First, there are known unknowns: even when further factors are known, it is not always possible to calcu-

[11] Thus, our approach diverges from Strevens' (2008, Chapter 8). According to him, idealisations work by deliberately misrepresenting non-difference makers by taking a parameter representing them to an extremal value. Using limit keys allows distortions to accurately represent systems even where they *do* make a difference. In fact, they allow us to quantify the difference that they make by means of the size of the interval that results after applying the key.

late their effect explicitly. We know that the real slope is uneven in various ways and that this unevenness has an effect on the real skier's motion, but we cannot capture this effect mathematically. Nor can we calculate the effect of air resistance that crucially depends on the skier's shape, which we know not to be a rectangular block! And so on. So we cannot always explicitly specify the difference between a model and the target as we did in the last section; linear friction is a special case in that regard. Second, and worse still, there may be unknown unknowns: we may not know all the factors that influence a situation. For example, the skier may be subject to forces we don't know. Knowing all the relevant factors would require a God's eye perspective that mortal scientists don't have. The consequence of this is that in practice we cannot neatly quantify the differences between model and target, and we cannot rigorously prove that the model is a regular limit of sequence that contains the real-world target.

But it remains that when we reason using a limit key, we're relying on the existence of such a limit. In the abstract, such a key requires the following. We have a model with a particular exemplified feature (P). We assume that the model is the system that would result, were we to take all of the potentially relevant features of the target to a certain limit. As such, by exploiting this, we can reason from the fact that the model exemplifies P, and assuming that the model is the result of taking all of the relevant limits of the target, that the target's feature of interest will be within the interval $(P - \epsilon, P + \epsilon)$ around the feature P exemplified by the model (where ϵ will depend on the limit in question) . In terms of DEKI, Q is 'being in the interval $(P - \epsilon, P + \epsilon)$', and Q is imputed to the target. Now, whether or not the result of this reasoning, i.e. whether target's feature of interest is in this range, is true will depend on whether it is the case that by taking all the limits of features in the target we will in fact arrive at the model in question. And this is usually not the sort of thing that admits mathematical proof.

Does the fact that we cannot prove that the limit exists pull the rug from underneath limiting reasoning? For those who require mathematical proofs, yes. But there are rarely, if ever, mathematical proofs backing the successful application of a model to the world.[12] What scientists will do in this situation is to form a qualitative judgement against their background knowledge. They will take into account everything they know about forces and their effect on bodies, and they will make a qualitative estimate of the magnitude that this effect will have on the skier. This will give them an interval $(x_{\text{model}}(t) - \epsilon^e, x_{\text{model}}(t) + \epsilon^e)$, where the superscript 'e' stands for 'estimate', of which they will be willing to say that the real position of the skier will lie in that interval given everything they know about forces. This defines a feature Q^e that they can then impute to the target.

Limits have not become obsolete. The justification for imputing Q^e rests on the belief that a limit exists and that the model function is only so far away from it. Let us spell this out in more detail. Meet an old friend: Laplace's Demon

[12] And there are good reasons to doubt that we should expect there to be such proofs. Whether or not a model is an accurate representation depends on features beyond the model: it depends on the nature of the target system in question. As such, whilst we may be able to prove that if the target is such that by taking its relevant features to the limit we arrive at the model, then the model will allow us to reason successfully about the target, the antecedent of this conditional isn't the sort of thing that admits mathematical proof.

(Laplace 1814). The Demon knows all the forces and can write down the true position function $x_{\text{skier}}(t)$ of the skier. This function will depend on a myriad of parameters. The claim that scientists—mostly implicitly—rely on is that if the Demon now took all of the parameters in $x_{\text{skier}}(t)$ to their values in the model, that limit would turn out to exist and to be regular. That is, they assume $\lim x_{\text{skier}}(t) = x_{\text{model}}(t)$, where we write 'lim' (without subscripts) to indicate that the limit is taken for *all* parameters. Of course, $\lim x_{\text{skier}}(t) = x_{\text{model}}(t)$ is not provable, not least because human scientists, lacking the powers of the Demon, don't have access to $x_{\text{skier}}(t)$. It is a transcendental assumption in the sense that it must be made for it to be possible to apply the model using a limit key even though the assumption cannot be proven. But it is an assumption that scientists must make if they are to assume that the model is informative about the target (through a limit key). If the limit does not exist, or if it is singular, then there is no reason to assume that the target behaves like the model, even if the model's parameter values are close to the target's parameter values.

Carpets and ski jumpers are toy examples. But the same inferential patterns are at work in 'real' applications. Consider the Newtonian model of a planet's orbit. The model involves scientists imagining the following fictional scenario: two perfect spheres, both with a homogeneous mass distribution, are placed in otherwise empty space. One is much more massive than the other, and the only force acting on the spheres is the gravitational attraction between them. Combining these assumptions with Newton's second law, assuming that the heavier sphere is at rest, and letting $\vec{x}$ be the vector pointing from the centre of the heavier sphere to the centre of the lighter sphere, gives an equation of motion for the planet in the model: $\ddot{\vec{x}} = -Gm_s\vec{x}/|\vec{x}\,|^3$, where m_s is the mass of the heavier sphere, and G is the gravitational constant. The trajectory $\vec{x}_{\text{model}}(t)$ of the model planet is the solution of this equation.

This equation of motion isn't the exact equation of motion governing the actual planet: even supposing that Newtonian mechanics were correct, the actual force that determines how a planet moves includes forces beyond its gravitational interaction with the sun. So we have an exemplified feature of a model, $\vec{x}_{\text{model}}(t)$, which we know doesn't match any actual feature of the target. What, then, does the motion of model-planet tell us about the motion of a real planet? The answer, we submit, is provided to us by a limit key. We should think of the actual trajectory $\vec{x}_{\text{planet}}(t)$, available to the Demon but not to us, as being such that if the Demon took all the parameters in $\vec{x}_{\text{planet}}(t)$ to limits corresponding to their value in the model—presumably most of them will be taken to zero given they don't appear in $\vec{x}_{\text{model}}(t)$—then the Demon would find that $\lim\vec{x}_{\text{target}}(t) = \vec{x}_{\text{model}}(t)$. If we combine this result with the assumption that the actual value of these parameters in the real world are not too far away from their values in the model, we can infer that the model trajectory is not too far away from the real trajectory.[13]

[13] Here we state the model-target relationship in terms of the model being 'close' to the real system, as standardly presented in physics. As noted above, limit keys obviously cover such cases, but they're not restricted to situations where the model is 'close' to the target. They just require that there be the right kind of systematic relationship between the parameter values and trajectory.

This kind of reasoning has been incredibly successful throughout the history of physics, and indeed engineering. From planetary motion to rocket launches, it has worked successfully in countless applications. This lends credibility to the use of limit keys in mechanics, and it makes scientists confident that limit keys will also work in future applications. It is important to realise, however, that inductive support for limit reasoning does not 'prove' the method right. In fact, scientists have worried about these limits time and again and delimiting the scope of their successful use has been a scientific endeavour in its own right. As an example, consider Poincaré's study of the role of initial conditions. Among the parameters that $\vec{x}_{\text{planet}}(t)$ contains are the planet's position and momentum at a certain initial time t_0. This is because Newton's equation of motion tells us where a planet is at a later time $t > t_0$ only if we specify the planet's position and momentum at some initial time. This specification is the planet's *initial condition*. In practice scientists can only ever specify an approximate initial condition because it's impossible to measure the condition with absolute precision.

Limit reasoning then would say that if the initial condition in the model is sufficiently close to the initial condition of the real planet, then the model-trajectory is sufficiently close to the real planet's trajectory (the comment in footnote 13 applies again here). Scientists took this assumption for granted until Poincaré showed that it was not true in general. Poincaré studied what is now known as the three-body-system, which is exactly like the Newtonian model except that it has a third sphere in it. If you want an interpretation, you can think of these three spheres as the sun, the earth, and the moon. What Poincaré found was that the three-body-system exhibits what is now known as sensitive dependence on initial conditions: even if two initial conditions are arbitrarily close, their trajectories can diverge. This effect is now also known as *chaos*.[14] This means that the limit does not exist and hence the model cannot be equipped with a limit key. This has far reaching consequences. Specifically, it means that Newton's model cannot be equipped with limit key and be expected to provide true results concerning a planet's trajectory, at least not universally and unrestrictedly. What exactly the restrictions are is a question that is discussed in the discipline of chaos theory. The details are beyond the scope of this paper, but one of the crucial results is that in contexts like the ones that Poincaré considered a limit key can be expected to deliver correct results only for finite time spans. So chaos theory tells us that the transcendental assumption is justified only for finite times.

And questions about limits go beyond initial conditions. What happens if the dynamics of the target system is different from the dynamics of the model in certain respects? This question promoted a study of what is now known as structural stability, which continues to date.[15] So the study of the boundaries of limits

[14] For a discussion of Poincaré's discovery of sensitive dependence on initial conditions see Parker 1998 and for discussion of the implications of chaos for predictability see Werndl 2009. For accessible introductions to chaos see, for instance, P. Smith 1998 and L.A. Smith 2007. For an advanced discussion see, for instance, Lichtenberg and Liebermann 1992.

[15] For technical discussion of results see Pilyugin 1991. Frigg *et al.* 2014 provide an accessible introduction and a discussion of philosophical consequences.

is not only a philosophically interesting issue; it is also a field of active scientific research.

6. Conclusion

Limit keys provide a concrete example of the sort of keys that the DEKI account of scientific representation urges we should focus on when investigating what our scientific models tell us about the world. Understanding how they work contributes to our broader understanding of scientific representation, and indeed the epistemic value of idealisation. Moreover, as demonstrated by the previous discussion, by requiring that we specify the key, thinking about (at least some) models in physics through the lens of DEKI helps us understand what sort of methodological assumptions underpin the use of those models. In order to understand how such models work, we have to pay careful attention to which features of a model are exemplified, and which features of its target are taken to which limit.

This lesson generalises to other, more philosophically contentious, models. For example, the Ising model of ferromagnetism invokes the thermodynamic limit, and is thus set on an infinite lattice (Baxter 1989). Given that its target systems—iron bars for example—do not consist of an infinite number of particles, how should we understand the idealisation present in the model? In this case, the problem is particularly pressing since the model in question underpins much of our current understanding of phase transitions. In the case of (the original interpretation of) the Ising model, the phase transition consists in an iron bar shifting between ferromagnetic and paramagnetic phrases, a transition which is understood as being represented by the occurrence of a non-analyticity in the model's free energy function. Taking the lattice to the infinite limit is *necessary* for the model to exhibit such a transition: for mathematical reasons, a non-analyticity cannot occur in the free energy function of a system with a finite particle number, and hence phase transitions—defined as non-analyticities—cannot occur in systems with finitely many particles. For this reason, physicist David Ruelle says that phase transitions only occur in systems that are 'idealized to be actually infinite' and that this 'idealization is necessary' (2004: 2).

In the DEKI framework, the way of analysing what the model tells us about actual, finite systems, requires specifying a key linking an exemplified feature of the model with a purported feature of the target. As such, we need to specify which feature of the target we're interested in, and how it's related to the relevant feature of the model. There are two available options. The first option is to take the relevant feature of the model to be the non-analyticity of the free energy function; in which case we are in a situation where we have a sequence of systems, each a finite lattice lacking such a feature, and an infinite model at the limit of such a sequence, having such a non-analyticity. Such a position is advocated, for instance, by Batterman (2001, 2011) who argues that the infinite model is different from the finite systems and that phase transitions are therefore emergent phenomena. Under this interpretation we have an example of a singular limit, and, as argued above, we cannot reason about the target based on the limit-key.

An alternative approach is recommended by Butterfield (2011) who argues that the relevant feature is not the non-analyticity of the free-energy function, but rather the free-energy function itself (or more specifically, the magnetisation of the lattice, which is the partial derivative of the free energy with respect to the external field).[16] In this case, if we again consider a sequence of lattices, we have a sequence of free-energy functions that converges pointwise to the free-energy function of the model (this is despite the fact that each of the free-energy functions on finite lattices is analytic, and the model's free-energy function is not). In which case we can employ the limit key strategy which we discussed in the last section.

Which of these points of view is correct is a deep question in the foundations of physics that we cannot address in this paper. Our aim here is a different one, namely to show that in order to reason using the limit key, the model must exemplify a feature that is the *regular* limit of a target-feature. Where an exemplified feature is like this, the key allows us to export a feature from the model to the target that the latter actually has. Conversely, if the exemplified feature is not like this, using a limit key will make the model an inaccurate representation.

This provides two general morals. First it demonstrates that properly understanding these cases of model-based science requires paying careful attention to which features of the models are exemplified, and which specific features of the target system are taken to which limit. The discussion of the Ising model generalises. Choosing a particular feature of the target system to focus on, and constructing a model that takes it to the limit in the right way, is a significant aspect of scientific modelling. Understood in the way we're urging, it is paramount that any model employing extremal features is evaluated carefully in terms of limits, and of how those limits are constructed. Second, and more generally, it demonstrates that limit keys provide concrete examples of the keys invoked in the DEKI account of scientific representation, thereby illuminating how it is to be explicated in practical applications. As applied to models that are idealised in the sense discussed here, this also demonstrates how idealisation—understood as the, sometimes radical, distortion of a relevant feature of a target—can play a positive epistemic role, despite, or even better, in virtue, of that distortion.[17]

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[16] Closely related points are made by Norton 2012. For a discussion see Palacios 2019.

[17] Thanks to Alberto Voltolini for inviting us to be part of this special issue, and to Ashton Green and David Lavis for helpful comments on an earlier draft. Thanks also to the audiences at the Universities of Bordeaux, Cambridge, and Oxford, as well as at the Philosophy of Science Association's Biennial Meeting (PSA18) in Seattle.

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